

RELATIVE ENTROPY FOR MAXIMAL ABELIAN SUBALGEBRAS OF MATRICES AND THE ENTROPY OF UNISTOCHASTIC MATRICES

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Dedicated to the memory of Professor Masahiro Nakamura

ABSTRACT. Let A and B be two maximal abelian $*$ -subalgebras of the $n \times n$ complex matrices $M_n(\mathbb{C})$. To study the movement of the inner automorphisms of $M_n(\mathbb{C})$, we modify the Connes-Størmer relative entropy $H(A|B)$ and the Connes relative entropy $H_\phi(A|B)$ with respect to a state ϕ , and introduce the two kinds of the constant $h(A|B)$ and $h_\phi(A|B)$. For the unistochastic matrix $b(u)$ defined by a unitary u with $B = uAu^*$, we show that $h(A|B)$ is the entropy $H(b(u))$ of $b(u)$. This is obtained by our computation of $h_\phi(A|B)$. The $h(A|B)$ attains to the maximal value $\log n$ if and only if the pair $\{A, B\}$ is orthogonal in the sense of Popa.

1. INTRODUCTION

In a step to introduce the notion of the entropy for automorphisms on operator algebras, Connes and Størmer defined in [2] the relative entropy $H(A|B)$ for finite dimensional von Neumann subalgebras A and B of a finite von Neumann algebra M with a fixed normal normalized trace τ .

After then, the study about the $H(A|B)$ is extended to two directions.

One development was started by Pimsner and Popa in [7] as the relative entropy $H(A|B)$ for arbitrary von Neumann subalgebras A and B of M , (see [4] for many interesting results in this direction).

The other was a generalization due to Connes in [1] by changing the trace τ to a state ϕ on M . He defined the relative entropy $H_\phi(A|B)$ for two subalgebras A and B of M with respect to ϕ .

We modify the Connes-Størmer relative entropy $H(A|B)$ and the Connes relative entropy $H_\phi(A|B)$ with respect to a state ϕ , and we introduce the corresponding two kinds of the constant $h(A|B)$ and

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$h_\phi(A|B)$. In the case where $A = M$, $h(M | B)$ is nothing else but the Connes-Størmer relative entropy $H(A | B)$. In general, $0 \leq h_\phi(A|B) \leq H_\phi(A|B)$, and if M is an abelian von Neumann algebra, then $h_\phi(A | B) = H_\phi(A | B)$ so that it coincides with the conditional entropy in the ergodic theory.

In this paper, we restrict our subjects to the maximal abelian subalgebras (abbreviated as MASA's) of the $n \times n$ complex matrices $M_n(\mathbb{C})$.

If A and B are two MASA's of $M_n(\mathbb{C})$, then there exists a unitary matrix u with $B = uAu^*$, which we denote by $u(A, B)$. Each unitary matrix u induces a unistochastic matrix $b(u)$, which is a typical example of a bistochastic matrix.

For a bistochastic matrix b , it is introduced in [9] the notion of the entropy $H(b)$ and the weighted entropy $H_\lambda(b)$ with respect to λ .

Here we show a relation between $h(A | B)$ for a pair $\{A, B\}$ of MASA's and the $H(b(u(A, B)))$. This is a consequence of our computation on $h_\phi(D | uDu^*)$, where ϕ is a state of $M_n(\mathbb{C})$ and D is the set of $n \times n$ diagonal matrices corresponding to the eigenvectors of ϕ . The $h_\phi(D | uDu^*)$ satisfies the following relation :

$$h_\phi(D | uDu^*) = H_\lambda(b(u)^*) + S(\phi|_D) - S(\phi|_{uD u^*}),$$

where $\lambda = \{\lambda_1, \dots, \lambda_n\}$ is the eigenvalues for ϕ and $S(\psi)$ means the entropy of a positive linear functional ψ .

As the special case where ϕ is the normalized trace, we have

$$h(A | B) = H(b(u(A, B))).$$

In the case of the Connes-Størmer relative entropy, for a given two MASA's A and B of $M_n(\mathbb{C})$, $H(A | B)$ is not equal to $H(b(u(A, B)))$ in general. For an example, see [6, Appendix].

The above results show that $h_\phi(A|B)$ for MASA's of $M_n(\mathbb{C})$ are determined by the entropy for the related unistochastic matrices in the sense of [9]. Also, the value $h(D | uDu^*)$ is depending on the inner automorphism Ad_u defined by the unitary u , and we can consider these values as a kind of conditional entropy for Ad_u conditioned by D .

Two MASA's A and B of $M_n(\mathbb{C})$ are *orthogonal* in the sense of Popa [8] if

$$A \cap B = \mathbb{C}1 \quad \text{and} \quad E_A E_B = E_B E_A = E_{A \cap B} = E_{\mathbb{C}1},$$

where E_A is the conditional expectation of $M_n(\mathbb{C})$ onto A such that $\tau \circ E_A = \tau$ for the normalized trace τ of $M_n(\mathbb{C})$. This means that

$$\begin{array}{ccc} B & \subset & M_n(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}1 & \subset & A \end{array}$$

is a commuting square in the sense of [3].

We have that a pair $\{A, B\}$ of MASA's is orthogonal if and only if $h(A|B)$ takes the maximal value, and then the value is $\log n$.

2. PRELIMINARIES

In this section, we summarize notations, terminologies and basic facts which we need later.

Let M be a finite von Neumann algebra, and let τ be a fixed faithful normal tracial state. By a von Neumann subalgebra A of M , we mean that A has the same identity with M . Let E_A be the τ -conditional expectations on A .

2.1. Relative entropy.

2.1.1. *Relative entropy of Connes-Størmer.* First, we review about the formulae of the relative entropy by Connes and Størmer in [2] (cf. [4]).

Let S be the set of all finite families (x_i) of positive elements in M with $1 = \sum_i x_i$. Let A and B be two von Neumann subalgebras of M .

The relative entropy $H(A|B)$ is

$$H(A|B) = \sup_{(x_i) \in S} \sum_i (\tau \eta E_B(x_i) - \tau \eta E_A(x_i)).$$

Here $\eta(t) = -t \log t$, $(0 < t \leq 1)$ and $\eta(0) = 0$.

Let ϕ be a normal state on M . Let Φ be the set of all finite families (ϕ_i) of positive linear functionals on M with $\phi = \sum_i \phi_i$.

The relative entropy $H_\phi(A|B)$ of A and B with respect to ϕ is

$$H_\phi(A|B) = \sup_{(\phi_i) \in \Phi} \sum_i (S(\phi_i|_A, \phi|_A) - S(\phi_i|_B, \phi|_B))$$

where $S(\psi|\varphi)$ is the relative entropy for two positive linear functionals ψ and φ , and $H_\tau(A|B) = H(A|B)$.

2.1.2. *Relative entropy of positive linear functionals.* After, we need the precise form $S(\psi|\varphi)$ in the case of finite dimensional algebras C , mainly the full matrix algebra $M_n(\mathbb{C})$ (the set of the $n \times n$ matrix $x = (x(i, j))_{ij}$ with $x(i, j) \in \mathbb{C}$).

We denote by Tr the canonical trace on C , that is, $\text{Tr}(p) = 1$ for every minimal projection $p \in C$.

Let ψ be a positive linear functional on C . We denote by Q_ψ the density operator of ψ , that is, $Q_\psi \in C$ is a unique positive operator with

$$\psi(x) = \text{Tr}(Q_\psi x), \quad (x \in C),$$

and the von Neumann entropy of ψ is given by

$$S(\psi) = \text{Tr}(\eta(Q_\psi)),$$

Let ψ and φ be two positive linear functionals on C . If $\psi \leq \lambda\varphi$ for some $\lambda > 0$, then the relative entropy of ψ and φ is given as

$$S(\psi, \varphi) = \text{Tr}(Q_\psi(\log Q_\psi - \log Q_\varphi)),$$

(cf. [4], [5]).

2.1.3. *Conditional relative entropy.* Let A and B be von Neumann subalgebras of M . We modify $H(A|B)$ and $H_\phi(A|B)$ and as a replacement of $H(A|B)$ (resp. $H_\phi(A|B)$) we define the following constant $h(A|B)$ (resp. $h_\phi(A|B)$).

The *conditional relative entropy* $h(A | B)$ of A and B conditioned by A is

$$h(A | B) = \sup_{(x_i) \in S} \sum_i (\tau\eta E_B(E_A(x_i)) - \tau\eta E_A(x_i)).$$

Let $S(A) \subset S$ be the set of all finite families (x_i) of positive elements in A with $1 = \sum_i x_i$. Then it is clear that

$$h(A | B) = \sup_{(x_i) \in S(A)} \sum_i (\tau\eta E_B(x_i) - \tau\eta(x_i)).$$

Let $S'(A) \subset S(A)$ be the set of all finite families (x_i) with each x_i a scalar multiple of a projection in A . Then by the same proof with in [7],

$$h(A | B) = \sup_{(x_i) \in S'(A)} \sum_i (\tau\eta E_B(x_i) - \tau\eta(x_i)).$$

Hence, if A is finite dimensional, we only need to consider the families consisting of scalar multiples of orthogonal minimal projections.

Let ϕ be a normal state of M .

The *conditional relative entropy* of A and B with respect to ϕ conditioned by A is

$$h_\phi(A|B) = \sup_{(\phi_i) \in \Phi} \sum_i (S(\phi_i | A, \phi | A) - S((\phi_i \circ E_A) | B, (\phi \circ E_A) | B)).$$

Assume that M is finite dimensional and that the density matrix of ϕ is contained in A . Let $\Phi(A) \subset \Phi$ be the set of all finite families (ϕ_i) whose density operators $(Q_i)_i$ are contained in A . Since the density matrix of $\phi_i \circ E_A$ is $E_A(Q_i)$, we have

$$h_\phi(A|B) = \sup_{(\phi_i) \in \Phi(A)} \sum_i (S(\phi_i | A, \phi | A) - S(\phi_i | B, \phi | B)).$$

2.1.4. Remark. We will study about $h_\phi(A|B)$ and $h(A | B)$ for von Neumann subalgebras A and B of a general finite von Neumann algebra M elsewhere. Here we just remark the following facts.

(1) If ϕ is the normalized trace τ , then by [4, Theorem 2.3.1(x)]

$$h_\tau(A|B) = h(A | B).$$

(2) It is clear that $h(M | B)$ is nothing else but the Connes-Størmer relative entropy $H(M | B)$, and we have the relation with Index by Pimsner-Popa [7].

(3) In general, $0 \leq h_\phi(A|B) \leq H_\phi(A|B)$. If M is an abelian von Neumann algebra, then $h_\phi(A | B)$ coincides with the conditional entropy in the ergodic theory by a similar proof as in [4, p. 158].

2.2. Unistochastic matrices and the entropy. When a matrix $x \in M_n(\mathbb{C})$ is given, we denote the (i, j) -component of x by $x(i, j)$. A matrix $b \in M_n(\mathbb{C})$ is called *bistochastic* if $b(i, j) \geq 0$ for all $i, j = 1, \dots, n$, $\sum_{i=1}^n b(i, j) = 1$ for all $j = 1, \dots, n$ and $\sum_{j=1}^n b(i, j) = 1$ for all $i = 1, \dots, n$. Let $\lambda = \{\lambda_1, \dots, \lambda_n\}$ be a probability vector.

The weighted entropy $H_\lambda(b)$ of a bistochastic matrix b with respect to λ and the entropy $H(b)$ for a bistochastic matrix b are given in [9] as the following forms, respectively :

$$H_\lambda(b) = \sum_{k=1}^n \lambda_k \sum_{j=1}^n \eta(b(j, k)),$$

and

$$H(b) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \eta(b(i, j)).$$

Let u be a $n \times n$ unitary matrix. The *unistochastic matrix* b defined by u is the bistochastic matrix given as

$$b(i, j) = |u(i, j)|^2, \quad (i, j = 1, 2, \dots, n).$$

3. RESULTS

Lemma 1. *Let A be a maximal abelian subalgebra of $M_n(\mathbb{C})$, and let $\{p_1, \dots, p_n\}$ be the minimal projections of A .*

(1) *If ψ is a positive linear functional of $M_n(\mathbb{C})$, then*

$$S(\psi|_A) = \sum_{j=1}^n \eta(\psi(p_j)).$$

(2) *If ϕ is a state of $M_n(\mathbb{C})$ and if $\phi = \sum_i \phi_i$ is a finite decomposition of ϕ into a sum of positive linear functionals, then*

$$\sum_i S(\phi_i|_A, \phi|_A) = - \sum_i S(\phi_i|_A) + S(\phi|_A).$$

Proof. (1) Since the density operator of $\psi|_A$ is written as $\sum_j \psi(p_j)p_j$, we have

$$S(\psi|_A) = \text{Tr}_A\left(\sum_j \eta(\psi(p_j))p_j\right) = \sum_{j=1}^n \eta(\psi(p_j)).$$

(2) We denote the density operator of $\phi_i|_A$ by Q_i and the density operator of $\phi|_A$ by Q , then

$$\begin{aligned} \sum_i S(\phi_i|_A, \phi|_A) &= \sum_i \text{Tr}(Q_i(\log Q_i - \log Q)) \\ &= - \sum_i S(\phi_i|_A) - \text{Tr}\left(\sum_i Q_i \log Q\right) \\ &= - \sum_i S(\phi_i|_A) + S(\phi|_A). \end{aligned}$$

□

Let ϕ be a state of $M_n(\mathbb{C})$. We number the set of the eigenvalues of Q_ϕ as $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}$. Let us decompose Q_ϕ into the form $Q_\phi = \sum_{i=1}^n \lambda_i e_i$, where $\{e_1, \dots, e_n\}$ is a family of mutually orthogonal minimal projections in $M_n(\mathbb{C})$, which we fix. Let D be the MASA generated by the projections $\{e_1, \dots, e_n\}$, which we denote by $D(\phi)$ when we need.

Let $\{e_{kl}\}_{k,l=1,\dots,n}$ be a system of matrix units of $M_n(\mathbb{C})$ such that $e_{ii} = e_i$ for all $i = 1, \dots, n$. We give the matrix representation for

each $x \in M_n(\mathbb{C})$ depending on this matrix units $\{e_{kl}\}_{k,l=1,\dots,n}$ so that $D = D(\phi)$ is the diagonal algebra. Let $u \in M_n(\mathbb{C})$ be a unitary, and let $b(u)$ be the unistochastic matrix defined by u . Under these situations, we have the following theorem.

Theorem 2. *Let ϕ be a state of $M_n(\mathbb{C})$, and let D be the diagonal algebra of $M_n(\mathbb{C})$. Let $u \in M_n(\mathbb{C})$ be a unitary, then*

$$h_\phi(D \mid uDu^*) = H_\lambda(b(u)^*) + S(\phi|_D) - S(\phi|_{uD u^*}).$$

Proof. First we remark that Q_ϕ is contained in D and $S(\phi) = S(\phi|_D)$. Using the matrix representation of the unitary u , the matrix representation of each $ue_j u^*$ is given by

$$ue_j u^* = \sum_{k,l} u(k,j) \overline{u(l,j)} e_{k,l} = (u(k,j) \overline{u(l,j)})_{k,l}.$$

Let $(\phi_i)_{i=1,\dots,n}$ be the positive linear functionals of $M_n(\mathbb{C})$ such that $Q_{\phi_i} = \lambda_i e_i$ for all i . Then $\phi = \sum_i \phi_i$ gives a finite decomposition of ϕ , and

$$\begin{aligned} \sum_j \eta(\phi_i(ue_j u^*)) &= \sum_j \eta(\lambda_i \mid |u(i,j)|^2) \\ &= \sum_j (\eta(\lambda_i) \mid |u(i,j)|^2 + \lambda_i \eta(|u(i,j)|^2)) \\ &= \eta(\lambda_i) + \lambda_i \sum_j \eta(|u(i,j)|^2). \end{aligned}$$

Hence by Lemma 1,

$$\begin{aligned} &\sum_{i=1}^n (S(\phi_i|_D, \phi|_D) - S(\phi_i|_{uD u^*}, \phi|_{uD u^*})) \\ &= - \sum_i \sum_j \eta(\phi_i(e_j)) + \sum_j \eta(\lambda_j) \\ &\quad + \sum_i \sum_j \eta(\phi_i(ue_j u^*)) - \sum_j \eta(\phi(ue_j u^*)) \\ &= \sum_i \sum_j \eta(\phi_i(ue_j u^*)) - \sum_j \eta(\phi(ue_j u^*)) \\ &= S(\phi|_D) + H_\lambda(b(u)^*) - S(\phi|_{uD u^*}). \end{aligned}$$

Thus

$$H_\phi(D \mid uDu^*) \geq H_\lambda(b(u)^*) + S(\phi|_D) - S(\phi|_{uD u^*}).$$

To prove the opposite inequality, assume that $(\phi)_{i \in I}$ be a given finite family which is contained in $\Phi(D)$. We denote Q_{ϕ_i} simply by Q_i .

Then

$$\lambda_l = \sum_{i \in I} Q_i(l, l), \quad \text{and} \quad Q_i(l, k) = 0 \quad \text{if} \quad l \neq k,$$

for all $l, k = 1, \dots, n$ and $i \in I$, and

$$\phi_i(ue_j u^*) = \sum_k Q_i(k, k) |u(k, j)|^2 \quad \text{and} \quad \phi_i(e_j) = Q_i(j, j),$$

for all $i \in I$ and $j = 1, \dots, n$.

Since $\eta(st) = \eta(s)t + s\eta(t)$ and $\eta(s+t) \leq \eta(s) + \eta(t)$ for all positive numbers s and t , by using that $\sum_j |u(k, j)|^2 = 1$ for all k , we have that

$$\begin{aligned} & \sum_i S(\phi_i |_{uD u^*}) - \sum_i S(\phi_i |_D) \\ &= \sum_i \sum_j \eta\left(\sum_k Q_i(k, k) |u(k, j)|^2\right) - \sum_i \sum_j \eta(Q_i(j, j)) \\ &\leq \sum_i \sum_j \left\{ \sum_k \eta(Q_i(k, k)) |u(k, j)|^2 + \sum_k Q_i(k, k) \eta(|u(k, j)|^2) \right\} \\ &\quad - \sum_i \sum_j \eta(Q_i(j, j)) \\ &= \sum_i \sum_k \eta(Q_i(k, k)) \left(\sum_j |u(k, j)|^2 \right) + \sum_j \sum_k \left\{ \left(\sum_i Q_i(k, k) \right) \eta(|u(k, j)|^2) \right\} \\ &\quad - \sum_i \sum_j \eta(Q_i(j, j)) \\ &= \sum_i \sum_k \eta(Q_i(k, k)) + \sum_j \sum_k \lambda_k \eta(|u(k, j)|^2) - \sum_i \sum_j \eta(Q_i(j, j)) \\ &= H_\lambda(b(u)^*) \end{aligned}$$

These imply that by Lemma 1,

$$\begin{aligned} & \sum_{i \in I} \{S(\phi_i |_D, \phi |_D) - S(\phi_i |_{uD u^*}, \phi |_{uD u^*})\} \\ &= - \sum_i S(\phi_i |_D) + S(\phi |_D) + \sum_i S(\phi_i |_{uD u^*}) - S(\phi |_{uD u^*}) \\ &\leq H_\lambda(b(u)^*) + S(\phi |_D) - S(\phi |_{uD u^*}) \end{aligned}$$

Hence we have always that

$$h_\phi(D |_{uD u^*}) \leq H_\lambda(b(u)^*) + S(\phi |_D) - S(\phi |_{uD u^*}),$$

and

$$h_\phi(D|uD u^*) = H_\lambda(b(u)^*) + S(\phi|_D) - S(\phi|_{uD u^*}).$$

□

Let D be the algebra of diagonal $n \times n$ matrices. Let ϕ be a state of $M_n(\mathbb{C})$, and let v be a unitary such that $v D v^* = D(\phi)$. Let ϕ_v be the state given by the inner perturbation by v : $\phi_v(x) = v x v^*$. Then we have the following relation between $h(A|B)$ and the set of $\{h_\phi(A|B)\}_\phi$.

Corollary 3. *Let D be the algebra of diagonal $n \times n$ matrices, and let $u \in M_n(\mathbb{C})$ be a unitary. Then*

$$\begin{aligned} h(D | u D u^*) &= H(b(u)) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \eta(|u(i, j)|^2) \\ &= \max_{\phi} h_{\phi_v}(D | u D u^*), \end{aligned}$$

where the maximum is taken over all states ϕ of $M_n(\mathbb{C})$.

Proof. The first equality is clear because $h(D | u D u^*) = h_\tau(D | u D u^*)$ and $H_\lambda(b(u)^*) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \eta(|u(i, j)|^2)$ where the eigenvalues for τ is $\lambda = \{1/n, \dots, 1/n\}$. We denote the minimal projections of D by $\{e_1, \dots, e_n\}$. Let ϕ be a state, and let $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be eigenvalues of Q_ϕ . We decompose $Q_\phi = \sum_{i=1}^n \lambda_i p_i$ by mutually orthogonal minimal projections $\{p_1, p_2, \dots, p_n\}$. Let v be a unitary with $v e_i v^* = p_i$ for all i . Then

$$Q_{\phi_v} = \sum_i \lambda_i e_i = v^* Q_\phi v.$$

That is, $D(\phi_v) = D$ and $S(\phi_v) = S((\phi_v)|_D)$. Since

$$S(\phi_v) \leq S((\phi_v)|_{uD u^*})$$

(cf. [4, Theorem 2.2.2 (vii)]), we have by Theorem 2 that, for the unistochastic matrix $b(u)$ defined by u ,

$$\begin{aligned} h_{\phi_v}(D | u D u^*) &= H_\lambda(b(u)^*) + S((\phi_v)|_D) - S((\phi_v)|_{uD u^*}) \\ &\leq H_\lambda(b(u)^*) \\ &\leq H(b(u)^*) = H(b(u)) = h(D | u D u^*). \end{aligned}$$

□

As another easy consequence of Theorem 2, by the property of the normalized trace τ , we have the following statement for general MASA's A and B :

Corollary 4. *Let A and B be maximal abelian subalgebras of $M_n(\mathbb{C})$. Then*

$$h(A \mid B) = H(b(u)) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \eta(|u(i, j)|^2)$$

where the $(u(i, j))_{ij}$ for $u = u(A, B)$ is given by the matrix units whose minimal projections generates A .

Corollary 5. *Let $\{A_0, B_0\}$ be a pair of maximal abelian subalgebras of $M_n(\mathbb{C})$. Then $\{A_0, B_0\}$ is an orthogonal pair if and only if*

$$h(A_0 \mid B_0) = \log n = \max h(A \mid B),$$

where the maximum is taken over the set of a pair $\{A, B\}$ of maximal abelian subalgebras of $M_n(\mathbb{C})$.

Proof. Let $u \in M_n(\mathbb{C})$ be a unitary with $uAu^* = B$. By a characterization of Popa([8]), a pair $\{A, B\}$ of MASA's is orthogonal if and only if $\tau(au bu^*) = 0$ for all $a, b \in A$ with $\tau(a) = \tau(b) = 0$. This implies that $\{A, B\}$ is orthogonal if and only if

$$|u(j, k)| = 1/\sqrt{n} \quad \text{for all } j, k.$$

Hence we have the conclusion. \square

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